ESTUDOS RBEP

The comprehension of numerical relationships in the learning of fractions: a comparative study with Brazilian and Portuguese children

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Abstract

The understanding of rational numbers is one of the major conceptual challenges faced by students in mathematics learning in basic education. Regarding fractions, to establish the inverse relationship between the numerator and the denominator becomes a key issue in concept formation. The main goals of this study were: to understand if the inverse relationship between smaller quantities than the unit in quotient and part-whole situations influences the learning of fractions; and to compare Brazilian and Portuguese students' comprehension of the inverse relationship between quantities in fraction problems, using quotient and part-whole situations. The results indicate that students have a better grasp of the inverse relationship between the quantities in the quotient situation and also showed that Portuguese students performed significantly better than Brazilian students in the both types of situations. The discrepancy in student performance can be explained by differences in the curricula of mathematics in grade four in these countries. Implications for the teaching of mathematics in these two countries were also discussed.

Keywords: rational numbers; fractions; inverse relation.

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Resumo
A compreensão das relações numéricas na aprendizagem das frações: um estudo comparativo com crianças brasileiras e portuguesas

A compreensão dos números racionais é um dos maiores desafios conceituais enfrentados pelos estudantes na aprendizagem matemática durante a educação básica. No que diz respeito às frações, estabelecer a relação inversa entre o numerador e o denominador torna-se uma habilidade fundamental na construção do conceito. Os objetivos deste estudo foram: verificar como a compreensão da relação inversa entre quantidades menores do que a unidade, apresentadas nas situações de quociente e parte-todo, influencia na aprendizagem das frações; e perceber se existe diferença no desempenho entre alunos brasileiros e portugueses quanto à compreensão da relação inversa entre quantidades em problemas de fração. Os resultados indicam que os estudantes apresentam uma melhor compreensão da relação inversa entre quantidades na situação quociente e apontam que os desempenhos dos estudantes portugueses são significativamente melhores do que os dos estudantes brasileiros, nos diferentes tipos de situação. A discrepância no desempenho dos estudantes pode ser explicada pelas diferenças nos programas curriculares de matemática no quarto ano nesses países. Implicações no ensino da matemática nesses dois países foram discutidas.

Palavras-chave: números racionais; frações; relação inversa.

Introduction

The understanding of fractions is based on logical relationships that are associated with the idea of quotient or magnitude. The inverse relationship between quantities is a fundamental aspect of the conceptual understanding of fractions, which means that when a whole is split into equal parts, the more parts will correspond to smaller parts. Authors have suggested that comprehending this inverse relation requires the reorganization of numerical knowledge (Stafylidou; Vosniadou, 2004), due to the need to grasp that the properties of integers do not define numbers in general (Jordan *et al.*, 2013), and that understanding their magnitude requires the recognition of an infinite quantity of numbers between two fractions (Vamvakoussi; Vosniadou, 2010).

A broad and diverse set of studies has contributed to the conceptual complexity of fractions and their impact on mathematical learning (Stafylidou; Vosniadou, 2004; Mamede; Nunes; Bryant, 2005; Hecht; Vagi; Torgesen, 2007; Nunes; Bryant, 2008; Siegler; Thompson; Schneider, 2011). However, there are still different reasons why learning fractions represent a challenge for some students.

One of these reasons relates to symbolic representation, which involves two integers, a and b, (where $b \neq 0$), indicating the fraction $^{a}/_{b}$. In this case, Nunes and Bryant (2015) claim that representing one number using two numerical fields may induce the children to only regard one of these numbers or that they might not understand that there is an inverse relationship between the numerator and the denominator. Another reason relates to the fact that the ordering and the equivalence relationships work differently in the two numerical fields (Behr *et al*., 1984; Nunes; Bryant, 2015).

A third reason concerns the significance that the numerator and denominator have in relation to the various situations in which fractions are used (Behr *et al*., 1984; Kieren, 1993; Nunes *et al*., 2004). Regardless of the type of difficulty that exists in learning fractions, They often continue into higher education (Stafylidou; Vosniadou, 2004; Vamvakoussi; Vosniadou, 2004).

In spite of the many existing studies into the comprehension of logical relationships and the conceptual knowledge of fractions, they are still a matter of concern to researchers and educators, mainly in terms of what children can learn about fractions and how this learning occurs. Thus, the aims of this study are: to see how the inverse relation between quantities smaller than the unit, presented in quotient and part-whole situations, influences the learning of fractions; and to investigate whether there is any difference in the performance of Brazilian and Portuguese in relation to their understanding of the inverse relation between quantities represented by fraction. Before generalizations could be established, the cultural proximity and language similarity were relevant aspects considered when comparing the performance of children from these two countries. A decision was made to investigate the logical relation that is presented in the learning of fractions, which was highlighted in the literature as an essential concept for algebra and mathematics at more advanced levels (Nunes; Bryant, 2008).

This article comprises three sections: the first presents a numerical knowledge literature review; the second describes the method of this study; and the final section discusses the results and educational implications of its empirical findings.

Relevance and challenges in learning fractions

Research on numerical knowledge (Behr *et al*., 1984; Kieren, 1993; Stafylidou; Vosniadou, 2004; Nunes; Bryant, 2008; Siegler; Thompson; Schneider, 2011; Hallett *et al*., 2012; Jordan *et al*., 2013) revolves around the idea that rational numbers implicate more complex notions than integers.

Learning fractions has an important role to play in the field of mathematics, which consists of representing quantities that cannot be described with integers and the relationships between quantities. Quantities smaller than one unit can arise in a division situation or a measurement situation (Behr *et al*., 1984; Kieren, 1993; Nunes *et al*., 2004).

A significant number of teaching approaches introduce fractions using part-whole situations. In these situations, the denominator of a fraction indicates the number of equal parts on which a whole was divided and the numerator indicates the parts considered. For example, the $\frac{2}{3}$ fraction of a cake is understood as a cake that is divided into three equal parts, and two of these parts are taken into consideration. The possibility of ordering fractions by magnitude tends to elude many children because they think that the larger the fractions values, the greater the quantity it represents (Nunes; Bryant, 2015). In the quotient situation, the numerator represents the amount to be shared and the denominator represents the number of beneficiaries. For example, $\frac{2}{3}$ represents that two chocolate bars were distributed among three children but also that each child received $\frac{2}{3}$ of the chocolate bar. The problems of the part-whole situation involve a multiplicative relation between two quantities of the same magnitude and the use of fair division or partitioning of a whole into equal parts (Vergnaud, 1983; Kieren; 1993). The problems regarding the quotient situation involve a multiplicative relation between two quantities of different magnitudes and the use of one-to-many correspondence, which may or may not involve division.

The relationships established between the numerator and the denominator involve complex cognitive skills for dealing with the consequences of changes in the size of each of them: (a) for the same denominator, the larger the numerator, the larger the fraction, (for example, $\frac{2}{4} < \frac{3}{4}$ expresses a direct relationship); (b) for the same numerator, the larger the denominator, the smaller the fraction, (for example, $\frac{1}{2} > \frac{1}{3}$, referring to the inverse relationship); and (c) numerator and denominator are different, (for example, $\frac{2}{3}$ and $\frac{4}{5}$ involve proportional relationships) (Behr *et al.*, 1984; Nunes; Bryant, 2008). Literature has been suggesting that children's grasp of the inverse relation between quantities is facilitated in the quotient situation (Mamede, 2007; Mamede; Nunes; Bryant, 2005; Streefland, 1997). Nunes *et al.* (2004) argue that this is due to the fact that the numerator and denominator are different types of variables.

Understanding the inverse relationship between quantities smaller than the unit

The conceptual difference between fraction situations was a source of inspiration for drawing comparisons between children's understanding of the equivalence and ordering of fractions using the quotient and part-whole situations. Behr *et al.* (1984) highlight that understanding the size represented by a fraction is fundamental for the development of underlying conceptual abilities in terms of ordering and equivalence of fractions.

Ordering fractions that are smaller than the unit involve the knowledge of the inverse relationship between the numerator and the denominator, that is, when a whole is divided into other equal parts the parts will be smaller and that there is no compensation between the size and the number of parts. In its turn, the equivalence of fractions smaller than the unit involves understanding the inverse proportional relation, which implies that in order to double the number of parts, each part must be half of its original size to ensure that sizes are equivalent. Hence, to understand the equivalence of fractions requires establishing compensatory relationships between the area and the number of equal parts in which the unit was divided. Thus, it is necessary to understand that fractions, which refer to the same quantity, can be represented by different symbols, such as ½, 2⁄4, 3⁄6, and can be expressed by different names (one half, two quarters, three sixths, etc.), and can form equivalence classes (Behr *et al.*, 1983). A study performed by Mamede, Nunes and Bryant (2005) investigated the understanding of quantities represented by fractions in quotient, part-whole and fractional operator situations with 80 children aged six and seven years old, before receiving formal instruction on fractions in school. The results indicated that children performed better in the quotient than in the part-whole situation, both in ordering and equivalence problems of fractions, and performed similarly in the naming tasks. A study by Nunes and Bryant (2008) examined the comprehension of equivalence and ordering of children in their fourth and fifth graders from eight schools in England. The tasks involved comparing fractions, such as ½ and ⅓; ½ and 2⁄4; and without referring to a situational context, which demonstrated that the linguistic and numerical symbols have a supporting role to play in students' grasp of equivalence and ordering. In addition, the authors suggested that perception based on numeric symbols is not sufficient to understand equivalence and ordering in the context of rational numbers. Kieren (1988) had previously suggested that these representations play an amplifying role in natural and structuring abilities during activities, from which reasoning is driven.

The possibility of exploring significant contexts from daily life provides the opportunity to use problem-solving strategies (rather than memorized procedures), which can stimulate reasoning and communication, in addition to being a facilitator for learning mathematics (Behr; Post; Lesh, 1981). For researchers, correspondence is a problem-solving strategy that involves multiplicative relationships (Nunes *et al.*, 2010). Different models can be used for solving quantitative problems. Nunes and Bryant (2015) highlight three factors involved in solving problems and argue that each of these factors has an impact on the way we learn. These factors are as follows: reflective thinking, sociocultural interaction and use of learning tools. Reflective thinking consists of a mental activity on behalf of the student, which is expressed by imagining or relating ideas. Socio-cultural interaction advances the development of mathematical ideas by students, which occurs through their interactions with the environment. Finally, the use of learning tools can help develop strategies and procedures for solving problems. A study by Hecht, Vagi and Torgesen (2007) suggests that children with learning disabilities use inaccurate mental models as well as memorized and incorrect procedures to solve mathematical problems.

As different situations in which fractions are used involve distinct linguistic and contextual support, it becomes relevant to further explore how different situations in which fractions are used affect students' understanding of the concept.

Teaching fractions during the early years of basic education in Brazil and Portugal

The poor achievement in learning fractions can be associated with the approach to fractions in school, where mainly one of the meanings of fractions is explored (Kieren, 1993). Mathematics in the classroom often introduces fractions with models of geometric figures and with part-whole situations.

In Brazil, the National Curricular Parameters (PCN, *Parâmetros Curriculares Nacionais*) emphasize that the approach to rational numbers aims at leading students to realize that the natural numbers already-known are insufficient to solve certain problems (Brasil. MEC, 1997). It is a study that starts in the second cycle of elementary school and it is consolidated in the final two cycles of school. According to the PCN (Brasil. MEC, 1997), teaching rational numbers start with their recognition in daily life. Also according to PCN (Brasil. MEC, 1997), learning rational numbers implies challenging with ideas built by students about natural numbers. Therefore, it takes time and demands the right approach.

In Portugal, the Mathematics Program for Basic Education (PMEB, *Programa da Matemática no Ensino Básico*) includes rational numbers in the first cycle of basic education (Portugal. MEC, 2007a). Thus, it begins to guide the initial notion of fraction and its different meanings, simultaneously exploring the fractional and decimal representation and developing number sense. Initially, the PMEB emphasizes "[...] an intuitive approach from situations of equal sharing and equal division, is resorting to models and representation in fraction form" (Portugal. MEC, 2007a, p. 15). The PMEB highlights the development of three essential skills: problem-solving, mathematical reasoning and mathematical communication. The PMEB proposes that the problems serve as an application context for knowledge acquired previously; they also serve for building new knowledge. It also points to how thinking processes used in performing different types of mathematical tasks are vital to the students' grasp of the concept. Finally, it points to the relevance of students' ability to interpret and communicate their mathematical ideas orally and to represent them in written ways, as well as to understand the ideas expressed by others.

To improve the quality of mathematics in school, the National Council of Teachers of Mathematics (NCTM) established principles and standards for curriculum and evaluation to the subject. In view of the NCTM, numerical knowledge involves counting, quantities of comparison, and advancing the understanding of the structure of the numerical system of base ten. The document asserts the importance of understanding the learning process of mathematics and emphasizes that the compression of numbers becomes more complex when it involves fractions and whole number. This requires that the teaching of mathematics engage students in solving a variety of problems, not just as one of the goals of learning but as a major means of doing so. The principles and standards indicate that problem-solving develops mathematical reasoning and promotes communication and representation that are part of the process of learning mathematics (NCTM, 2000).

In short, one can say that the official documents of Brazil and Portugal include in their curriculum guidelines the perspective of the principles of the US program proposed by the NCTM (2000). The Portuguese curriculum has extended the time in which children's activities involving rational numbers are exposed. Brazilian curricular parameters emphasize the use of problem solving as a teaching strategy.

Brazilian and Portuguese researchers have broadened the study on teaching and learning rational numbers. During an exploratory study with ten and 11-years-old Portuguese students in their fifth grade, Ponte and Quaresma (2014) investigated the multiple representations of rational numbers with different uses and types of grandeur and showed that the comprehension of ordering and comparing fractions combines formal and informal reasoning. From a different study on fractions, performed by Mamede and Vasconcelos (2014) with Portuguese fourth graders, the existence of a relationship between students' performance in quotient and part-whole fraction situations, as well as ordering and equivalence relations, was found. The results showed a significant correlation between ordering and equivalence in both situations.

Research performed by Campos and Magina (2004) with 70 teachers of third and fourth graders demonstrated that teachers have an impression regarding their students' performance that differs from reality and tend to overestimate their ability. The results indicated that the teachers adequately conceptualized fractions in some situations, in spite of the fact that they misrepresented fractions and ratio. However, comparative studies investigating fractions are scarcer. In Brazil, the current study found one comparative study with Brazilian and Portuguese children, aged six and seven, which was performed by Dorneles, Mamede and Nunes (2008). The study investigated ordering, equivalence and naming of fractions in quotient and part-whole situations. The performance of the children indicated greater levels of success in fraction ordering and equivalence tasks in the quotient meaning than in the part-whole meaning. This suggests that they had some informal knowledge on the logic of fractions developed in their everyday lives, without any formal education. The authors suggest that this fact can be explained as a consequence of the earlier and more extensive childhood education in Portugal. The present study addresses the following questions: How do children understand the inverse relationship between quantities and fractions in part-whole and quotient situations? Are there any differences in the performance of Brazilian and Portuguese students when solving problems in quotient and part-whole situations? The similar language and the cultural proximity enabled the comparison of the performance of Brazilian and Portuguese students.

Method

Participants

Ninety Brazilian and 73 Portuguese school children participated in this study. The children were aged between nine and ten years and were in their fourth grade of basic education within the public school network of two cities, Porto Alegre, Brazil, and Braga, Portugal. None of the students, from Brazil or Portugal, had received any formal instruction on the conceptual knowledge of fractions. In order to avoid discrepancies in comparing the results, the subjects from the sample were defined according to these three aspects: (a) both schools were part of the public education network and had similar socioeconomic profiles; (b) data analysis did not include information of students over 11 years or those with special needs; and (c) there was a similar level of knowledge regarding whole numbers and operations.

Procedure

This study is part of the research project entitled "Different groups of children with difficulties learning mathematics: what are the commonalities?" which was submitted and approved by the Research Ethics Committee at Federal University of Rio Grande do Sul (UFRGS), Brazil. All the data used for this comparative study among Brazilian and Portuguese children were collected by an academic intern at the University of Minho, Braga, Portugal.

A questionnaire was applied in order to verify the understanding of the inverse relationship between quantities smaller than the unit. The questionnaire comprised 16 problems divided into four categories: (1) ordering problems in a part-whole situation; (2) equivalence problems in a part-whole situation; (3) ordering problems in a quotient situation; and (4) problems of equivalence in a quotient situation. All problems were adapted from the study of Mamede, Nunes and Bryant (2005). The participants were given a book containing the problems, organized in such a way as to contain one on each sheet and to alternate each type of problem.

The instrument was collectively applied by the researchers in both countries during class time. The full-time teachers were present during the application of the instrument, which lasted approximately 50 minutes. The instrument had a similar construction in both countries, with only a few adaptations in terms of language. In Portugal, the instrument was applied by two of the authors of this article.

All the problems were projected on slides in the classroom and each one of them was read aloud to the group by one of the researchers, in order to avoid any interference of students' different levels of reading skills. After reading it, the students individually attempted to solve the problem, writing on their own booklet. Two minutes were given to solve each problem. The students then proceeded to solve the next problem.

Solving problems focused on logical reasoning of numbers and operations involve skills to compare amounts, establish relationships between quantities and judge the relative value: "more than; less than; same quantity as". These specific problems do not require numerical computation or arithmetic abilities. However, they require a greater understanding of the: (1) inverse relationship between two variables; (2) partitioning; (3) one-to-many correspondence; (4) ordering; and (5) equivalence. The solution to the problems involved three multiple-choice alternatives, only one being correct.

Instruments

Students' understanding of the inverse relationship between quantities smaller than the unit in part-whole situations was evaluated. The part-whole situation problems involve comparing quantities and contexts, in order to understand the changes when the numerator is constant and the denominator is variable. The part-whole situation problems that comprise the instrument are presented in Table 1.

Table 1 – Examples of Problems Presented in the Part-whole Situation

Part-whole situation problems	
Ordering	Equivalence
Marco and Lara have identical pizzas. Marco cut his pizza into two equal slices and ate one slice. Lara cut her pizza into three equal slices and ate one slice. Did Marco eat more pizza, less pizza or the same amount of pizza as Lara? Explain your answer.	Rita and Olga have identical cakes. Rita split her cake into two equal pieces and ate one piece. Olga split her cake into four equal pieces and ate two pieces. Did Rita eat more cake, less cake or the same amount of cake as Olga? Explain your answer.

Source: adapted from Mamede, Nunes and Bryant (2005).

Students' understanding of the inverse relationship between quantities smaller than the unit in quotient situations was evaluated. The comparison contexts make it possible to notice that if an amount is shared among more people then each one will receive less chocolate, the comparison also involves the correspondence of one bar to one child. The quotient situation problems that comprise the instrument are presented in Table 2.

Table 2 – Examples of Problems Presented in the Quotient Situation

Quotient situation problems	
Equivalence	Ordering
Two girls share one pizza equally and four boys share two pizzas equally. Did each of the girls eat more pizza, less pizza or the same amount of pizza as each of the boys? Explain your answer.	Two girls split one cake equally and three boys split one cake equally. Did each girl eat more cake, less cake or the same amount of cake as each of the boys? Explain your answer.

Source: adapted from Mamede, Nunes and Bryant (2005).

Visual models were used to support contextual situations, as the problems used in the assessment tool were not familiar to any of the students from both countries and this unfamiliarity could interfere with the results. Figure 1 shows an example of the model used in the problems.

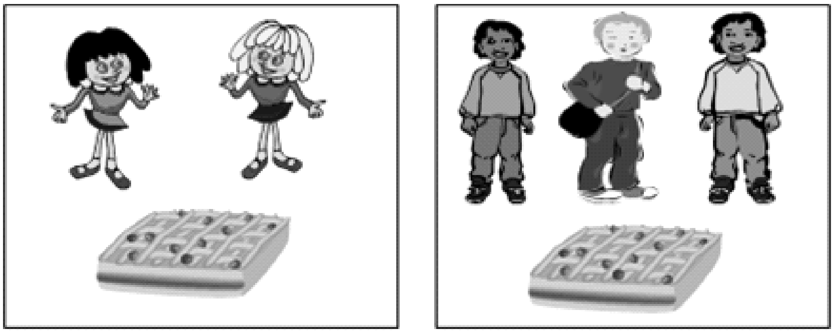


Figure 1 – Example of Figure Model Presented in the Quotient Situation Problem

Source: Mamede, Nunes and Bryant (2005).

Data analysis

Data analysis was carried out with the support of the software Statistical Package for Social Sciences (SPSS), version 18.

Results

In order to compare the performance of Brazilian and Portuguese students in solving problems, descriptives were carried out. A score of one was given for a correct answer and of zero for an incorrect answer. The results were primarily examined using the mean and the standard deviation of correct responses, depending on the problem type. These results can be seen in Table 3.

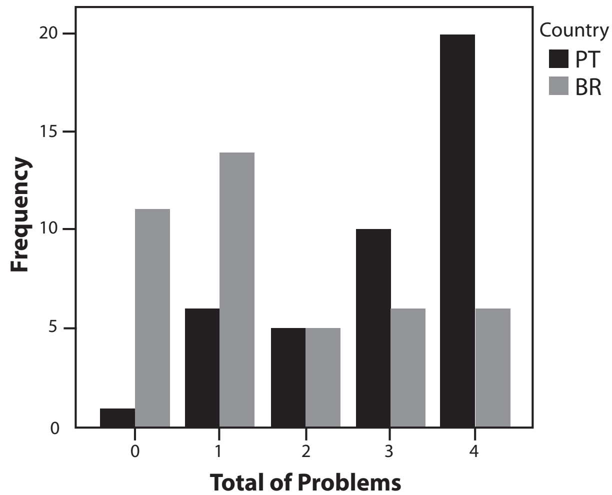
Table 3 – Mean (and Standard Deviation) of the Correct Responses Riven by Brazilian and Portuguese Students According to the Type of Problem

	Brazil (n = 90)		Portugal (n = 73)	
	Quotient	Part-whole	Quotient	Part-whole
Ordering	1,72 (1,40)	1,07 (1,14)	3,00 (1,19)	1,98 (1,47)
Equivalence	1,67 (1,43)	0,55 (0,86)	2,33 (1,41)	1,40 (1,31)

Source: Authors' elaboration.

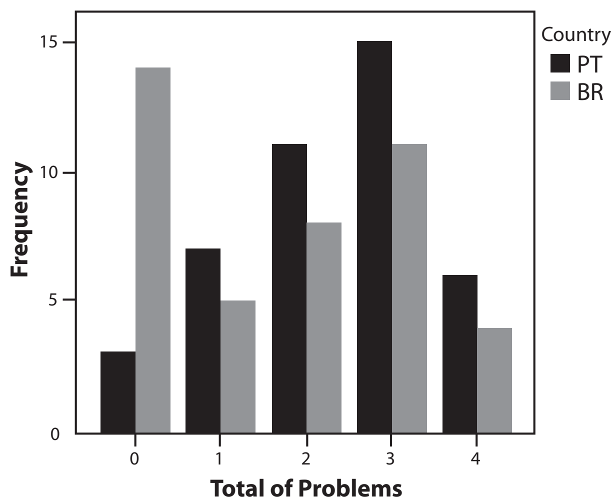
The results show that participants found quotient situation problems easier when compared with the part-whole situation. Both the Brazilian and the Portuguese students performed better in ordering than equivalence problems. In both situations, the Portuguese performed better in solving fraction problems than the Brazilian students. The distributions by country of children's performance in each type of fractions problem presented in quotient and part-whole situation are given in Graphics 1, 2, 3 and 4, respectively.

In the quotient situation, concerning ordering problems, 43.5% of the Brazilian and 83.3% of the Portuguese students solved at least two problems; 14.3% of the Brazilian and 47.6% of the Portuguese students answered all the problems correctly. In relation to equivalence, 42.7% of the Brazilian and 61.2% of the Portuguese students solved at least two problems; 9.5% of the Brazilian and 14.3% of the Portuguese students answered correctly all the problems.



Graphic 1 – Distribution by Country of Children's Correct Responses to Problems of Ordering of Fractions in Quotient Situation

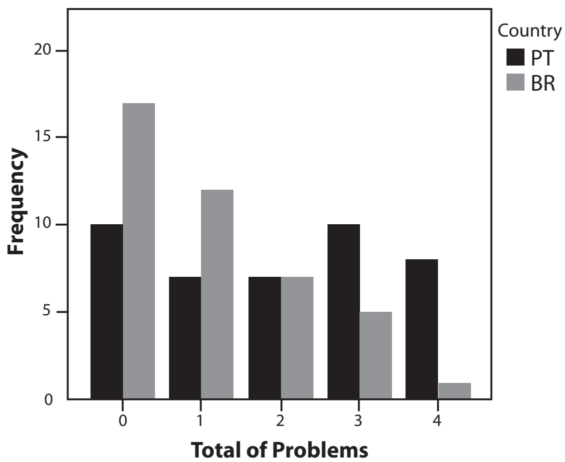
Source: Authors' elaboration.



Graphic 2 – Distribution by Country of Children's Correct Responses to Problems of Equivalence of Fractions in Quotient Situation

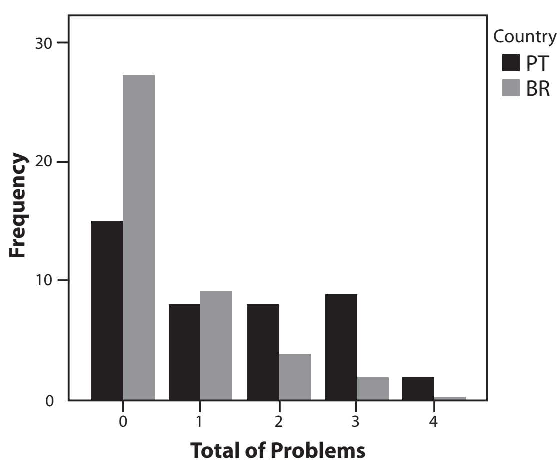
Source: Authors' elaboration.

In the part-whole situation, concerning ordering problems, 29% of the Brazilian and 59.5% of the Portuguese students solved at least two problems; 2.4% of the Brazilian and 19% of the Portuguese students correctly answered to all problems. In relation to equivalence problems, 14.3% of the Brazilian and 45.2% of the Portuguese students correctly responded at least to two problems; 4.8% of the Portuguese students were able to answer to all problems correctly. Nevertheless, no Brazilian student was able to do this.



Graphic 3 – Distribution by Country of Children's Correct Responses to Problems of Ordering of Fractions in Part-whole Situation

Source: Authors' elaboration.



Graphic 4 – Distribution by Country of Children's Correct Responses to Problems of Equivalence of Fractions in Part-whole Situation

Source: Authors' elaboration.

Students had a better performance in the quotient situation than in the part-whole situations has already been described in the literature (Mamede; Nunes; Bryant, 2005; Nunes *et al.*, 2007), which indicates that comprehending quantities related to division seems to be based on the acquired knowledge of fraction as a number.

In order to identify any differences between the groups of Brazilian and Portuguese students, the Mann-Whitney U test and Wilcoxon nonparametric T-test were applied, as data were not available. In order to analyze the students' reasoning and the strategies used in solving the problems, they were asked to justify their answers in writing. The Mann-Whitney U test showed a significant difference between the performance of the Brazilian and the Portuguese students ($p<0.001$). In quotient situations, the performance of the Portuguese students was significantly better in relation to ordering problems ($U = 403$; $W = 1.305$; $p<0.001$) and to equivalence problems ($U = 647$; $W = 1.550$; $p < 0.05$) than the performance of Brazilian students. This was also true in the part-whole situations, the performance of the Portuguese students was significantly better than that of Brazilian students concerning to ordering ($U = 573$; $W = 1476.5$; $p<0.05$) and equivalence problems ($U = 552.5$; $W = 1455.5$; $p<0.001$).

The significant discrepancy in the performance of the Brazilian and the Portuguese students suggests that students with a higher level of cognitive competency have a better comprehension of the inverse relation between quantities when resolving problems with fractions less than the unit. This disparity in performance can be explained by the differences in the mathematics curriculums provided during the fourth grade in these two countries. This finding is in agreement with the findings of the Hecht (1998) study, which suggests that the skills with fractions are influenced by the quality and amount of practice given to children in the classroom. The time that Portuguese children spend learning this content deserves a comment. It is worth remembering that childhood education in Portugal begins at age three, with 85% of Portuguese children starting from this age (Portugal. MEC, 2007); in Brazil, childhood education in public schools is offered from age five. In both countries, kindergarten is not compulsory.

Children's justifications for their responses were also analyzed in order to have an insight on their reasoning. Students' justifications were organised into five categories: inverse relationship, comprising all their explanations with a correct relation between the quantities producing a valid argument (e.g., "[. . .] because he divides his pizza into 2 equal parts and she divided hers into 4 equal parts and hers become smaller."); proportional reasoning, comprising a establishment of a proportional relation between the quantities of the problem, producing a valid argument (e.g. "they eat the same because there are two girls for one chocolate bar and the boys are the double of girls and they have the double of chocolate bars."); direct relationship, setting a direct relation between quantities (e.g. "he eats more because he has more cake, thus he eats more cake."); and inconclusive/invalid, comprising all the inconclusive, inappropriate, or blank explanations. Table 4 shows the percentage of justifications for each type of problem.

Table 4 – Percentage of the Types of Children's Justifications Given for each Type of Problem

	Quotient		Part-whole	
	Ordering (%)	Equivalence (%)	Ordering (%)	Equivalence (%)
	Students Brazilian (n = 90)			
Inverse relationship	37.8	32.2	11.1	7.8
Proportional reasoning	-	4.4	1.1	2.2
Direct relationship	25.6	24.4	25.6	60.0
Initial quantity	7.8	17.8	30.0	2.2
Inconclusive/invalid	28.9	21.1	32.2	27.8
Students Portuguese (n = 73)				
Inverse relationship	91.8	56.6	49.3	17.8
Proportional reasoning	1.4	15.1	2.7	6.8
Direct relationship	2.7	13.7	9.6	42.5
Initial quantity	1.4	6.8	26.0	8.2
Inconclusive/invalid	2.7	8.2	12.3	24.7

Source: Authors' elaboration.

Among the Portuguese children, the inverse relationship category has higher percentages, because the justification was used for most of the problems. This reasoning was based on valid arguments and on higher levels of understanding regarding the inverse relationship between quantities in fraction problems presented in quotient situations. These data seem to indicate that Portuguese students are able to produce mathematical reasoning and mathematical communication, as proposed by the PMEB. In quotient situation, the inverse relationship category also has higher percentages among Brazilian students. The higher performance in this situation did not occur by chance. However, the direct relationship and also inconclusive/invalid categories were used by Brazilian students and had a higher percentage in the part-whole situation. This result is consistent with the performance of these students, who experienced difficulty in part-whole situation. In Brazil, this difficulty can be attributed to the students' low usage and exploration of strategies that use reasoning to solving mathematical problems.

Discussion and conclusion

The results of this study indicate that the understanding of the inverse relationship between the numerator and the denominator is easier for children in the quotient situation than in the part-whole situation, even when taking into account the significant differences in the performances of Brazilian and Portuguese students. The result also suggests that the quotient and part-whole situations contribute in different ways to understand the inverse relationship between the numerator and denominator in learning fractions. Similar results were found by Mamede (2007) and by Mamede, Nunes and Bryant (2005), reinforcing the idea that the quotient situation favors a better comprehension of ordering and equivalence of fractions that relies on children's informal knowledge.

Moreover, the difficulty to establish the inverse relationship between the numerator and denominator in the part-whole situation is not a new finding. Similar results were found by Stafylidou and Vosniadou (2004), who showed that it is hard for students to grasp the relationship established between the numerator and the denominator of the fraction because they consider these two units independent quantities. Possible explanations for this result include the influence of experience gained with the integers and other limitations, such as the improper fraction $^4/_3$, which requires more complex quantitative reasoning when requested four pieces of a whole that were divided into three parts (Siegler; Thompson; Schneider, 2011).

Regarding ordering of fractions, the results indicate that students from both countries performed better in the quotient situation. It is possible that the comparison of two quantities of different nature could be favored as children easily rely on the use of one-to-many correspondence and distribution of items of receptors, which can easily be associated with a division but can also indicate an amount (Kieren, 1988).

Concerning equivalence of fractions, both the Brazilian and the Portuguese students' performances were significantly worse than it was in relation to ordering, which demonstrates that the difficulty in understanding the inverse relationship between the numerator and denominator. Similar results were found by Behr *et al*. (1984), who showed that it was hard for some of the children to understand the equivalence of fractions in the part-whole situation. The authors suggest that it is challenging for some children to establish a compensatory relationship between the numerator and the denominator of the fraction. This relation is based on inverse proportional reasoning, considering that doubling the parts means each part will be half the size of the whole.

Comparing the performances of Brazilian and Portuguese students, the Portuguese students were more accomplished in all types of problems. These results are similar to those reported by Dorneles, Mamede, and Nunes (2008), who found Portuguese students, aged six and seven who had not received formal education about fractions had a better performance than that of their Brazilian peers. These results converge with those of Hecht, Vagi and Torgsen (2007), who points out that the abilities related to the use of fractions can be influenced by the quality and quantity of classroom practice. This suggests that time of schooling may be an explanatory variable. Portuguese children begin early childhood education when they are three years old, and 85% of Portuguese children from that age are already in school; in Brazil, early childhood education is offered from age five in the public school system and it is not mandatory. Another associated fact may concern certain differences in the mathematics curriculum of the two countries. In Portugal, children have an informal contact with the concept of fraction in the second grade and formally in the third grade, according to the official curriculum guidance (Portugal. MEC, 2007a); in Brazil, in spite of the fact that the curriculum guidance (Brasil. MEC, 1997) indicates the formal contact in the fourth grade, teachers often avoid exploring fractions. Moreover, teachers' lack of conceptual knowledge of fractions may be related to the challenges they face using procedures on fractions without the proper grasp of the conceptual differences between the situations in which fractions are used. Alternatively, teaching situations become limited when operating with only one of the meanings of fractions (Hallett *et al.*, 2012).

The results of this research offer implications for teaching, especially the idea that children have informal knowledge about the logic of fractions from everyday experiences. This previous knowledge can be used to advance the understanding of the inverse relationship between quantities, thus supporting the learning of fractions.

This study highlights that children understand the relative nature of the fractions that underlie the ordering and equivalence of them. It can be beneficial to start teaching fractions with quotient situation problems, followed by part-whole situations. Nevertheless, teachers need to be aware of the fact that failing to exploit different situations related to fractions may compromise the understanding of rational numbers at various levels.

This study focused on the logical relations but not on the influence of the numerical representation of the fractions in the understanding of the inverse relationship between the numerator and denominator. More research needs to be carried on in order to explore how to advance the understanding of the inverse relation between fractions among children in the early years of basic education.

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